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MULTIPLICATIVITY FACTORS FOR C-NUMERICAL RADII

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## ABSTRACT

For matrices $A, C \in \underset{\sim}{C}_{n\times n}$, the C-numerical radius of $A$ is the nonnegative quantity

$$r_C(A) = \max\{|\mathrm{tr}(CU^*AU)| : U \text{ unitary}\} .$$

This generalizes the classical numerical radius $r(A)$. It is known that $r_C$ constitutes a norm on $\underset{\sim}{C}_{n\times n}$ if and only if $C$ is nonscalar and $\mathrm{tr}\, C \neq 0$. For all such $C$ we obtain multiplicativity factors for $r_C$, i.e., constants $\mu > 0$ for which $\mu r_C$ is sub-multiplicative on $\underset{\sim}{C}_{n\times n}$.


## 1. Introduction and statement of Main Results.

Let $\underset{\sim}{C}_{n\times n}$ denote the algebra of $n \times n$ complex matrices, and let

$$N : \underset{\sim}{C}_{n\times n} \to \underset{\sim}{R}$$

be a seminorm on $\underset{\sim}{C}_{n\times n}$, i.e., for all $A, B \in \underset{\sim}{C}_{n\times n}$ and $\alpha \in \underset{\sim}{C}$, let $N$ satisfy:

$$N(A) \geq 0,$$
$$N(\alpha A) = |\alpha| \cdot N(A) ,$$
$$N(A+B) \leq N(A) + N(B) .$$

If in addition $N$ is <u>positive definite</u>, that is,

$$N(A) > 0 \quad \text{for} \quad A \neq 0 ,$$

then following Ostrowski [9] we say that $N$ is a <u>generalized matrix norm</u>. Finally, if $N$ is also <u>(sub-) multiplicative</u>, namely

$$N(AB) \leq N(A)\, N(B) ,$$


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then $N$ is called a matrix norm. Hence, $N$ is a matrix norm if and only if it is an algebra norm on $\underset{\sim}{C}_{n\times n}$.

Given a seminorm $N$ on $\underset{\sim}{C}_{n\times n}$ and a fixed constant $\mu > 0$, then clearly

$$N_\mu \equiv \mu N$$

is a seminorm too. Similarly, $N_\mu$ is a generalized matrix norm if and only if $N$ is. In both cases, $N_\mu$ may or may not be multiplicative. If it is, then we call $\mu$ a multiplicativity factor for $N$.

The concept of multiplicativity factors was introduced by us in [4] where we proved the following:

THEOREM 1.1:

(i) [4, Theorem 3] Nontrivial, indefinite seminorms on $\underset{\sim}{C}_{n\times n}$ do not have multiplicativity factors.

(ii) [4, Theorem 4] If $N$ is a generalized matrix norm on $\underset{\sim}{C}_{n\times n}$, then $N$ has multiplicativity factors; and $\mu > 0$ is a multiplicativity factor for $N$ if and only if

$$\mu \geq \mu_N \equiv \max\{N(AB) : A,B \in \underset{\sim}{C}_{n\times n},\ N(A) = N(B) = 1\} \quad . \tag{1.1}$$

This result provides a better insight into the relation between positive-definiteness and submultiplicativity of seminorms on finite dimensional algebras.

One reason for introducing the idea of multiplicativity factors was to investigate the norm properties of C-numerical radii defined by us in [4] as follows: for given matrices $A,C \in \underset{\sim}{C}_{n\times n}$, the C-numerical radius of $A$ is the nonnegative quantity

$$r_C(A) = \max\{\mathrm{tr}|CU^*AU| : U \text{ n×n unitary}\} \quad ,$$

where * denotes the adjoint.

Evidently, for $C = \mathrm{diag}(1,0,\ldots,0)$, $r_C$ reduces to the classical numerical radius

$$(1.2) \qquad r(A) = \max\{|x^* A x| : x \in \underset{\sim}{C}^n, \; x^*x = 1\} \; .$$

hence $r_C$ is a generalization of $r$.

It is useful to recall now Lemma 9 of [3] which implies that $r_C$ is invariant under unitary similarities of $C$, i.e.,

$$r_{U^*CU}(A) = r_C(A), \; U \text{ unitary} \; .$$

Thus, we may assume that $C$ is upper triangular.

Regardless of the structure of $C$ we have:

THEOREM 1.2.

(i) (trivial) For any fixed $C$, $r_C$ is a seminorm on $\underset{\sim}{C}_{n\times n}$.

(ii) ([4, Theorem 2]; compare [8].) $r_C$ is a generalized matrix norm on $\underset{\sim}{C}_{n\times n}$, $n \geq 2$, if and only if

$$(1.3) \qquad C \text{ is a nonscalar matrix and } \operatorname{tr} C \neq 0 \; .$$

Theorems 1.1 (ii) and 1.2 (ii) yield now:

COROLLARY 1.1. For $n \geq 2$, $r_C$ has multiplicativity factors if and only if $C$ satisfies (1.3).

Theorem 4.1 of [5] (compare [4]) provides multiplicativity factors for all the C-numerical radii in Corollary 1.1, except for the case where $C$ has equal eigenvalues. In the present paper, we obtain multiplicativity factors for all $r_C$ satisfying (1.3) as well as improve our previous results as follows:

THEOREM 1.3. (Main Theorem.) Let $C = (\gamma_{ij}) \in \underset{\sim}{C}_{n\times n}$, $n \geq 2$, be a nonscalar, upper triangular matrix with $\operatorname{tr} C \neq 0$. Denote

$$\tau \equiv |\operatorname{tr} C| = \left|\sum_{j=1}^{n} \gamma_{jj}\right|, \quad \sigma \equiv \sum_{j=1}^{n} |\gamma_{jj}|, \quad \delta \equiv \max_{j,k} |\gamma_{jj} - \gamma_{kk}|,$$

(1.4)

$$\lambda \equiv \frac{\tau\delta}{4(1-1/n)\tau + 2\delta}, \quad \varphi \equiv \max\left\{\lambda, \frac{\delta}{4}\right\}$$

$$\omega \equiv \min\left\{\sum_{j=1}^{n}\sqrt{\sum_{k=j}^{n}|\gamma_{jk}|^2}, \ \sum_{k=1}^{n}\sqrt{\sum_{j=1}^{k}|\gamma_{jk}|^2}\right\}$$

$$\gamma \equiv \max_{j<k}|\gamma_{jk}|, \quad \rho \equiv \frac{\tau\gamma}{2\tau + 2\gamma}, \quad \nu \equiv \max\left\{\lambda, \frac{\tau\gamma - (n-1)\delta\gamma}{4\tau + 2\gamma}\right\} .$$

Then:

(i) If $C$ is normal (i.e., diagonal) with eigenvalues of the same argument, then any $\mu$ with

(1.5) $$\mu \geq \tau/\varphi^2 = \sigma/\varphi^2$$

is a multiplicativity factor for $r_C$.

(ii) If $C$ is normal, then $\mu$ is a multiplicativity factor for $r_C$ if

(1.6) $$\mu \geq \sigma/\lambda^2 \ .$$

(iii) If $C$ is nonnormal (i.e., nondiagonal) with equal eigenvalues, then any $\mu$ with

(1.7) $$\lambda \geq \omega/\rho^2$$

is a multiplicativity factor for $r_C$.

(iv) If $C$ is nonnormal and its eigenvalues are not all equal, then $\mu$ is a multiplicativity factor if

$$\mu \geq \omega/\nu^2 \ .$$

The proof of Theorem 1.3 is given in Section 2.

Evidently, Theorem 1.3 provides multiplicativity factors for all the C-radii which have such factors. Parts (i), (ii) and (iv) of the therem improve our results in Theorem 4.1 of [5], and part (iii) treats previously unattended cases.

The following table lists several typical examples:

| C | $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ | $\begin{pmatrix}5 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$ | $\begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}$ | $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ | $\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$ |
|---|---|---|---|---|---|---|
| Factors | none | none | $\mu \geq 96/25$ | $\mu \geq 64/3$ | $\mu \geq 9(1+\sqrt{2})$ | $\mu \geq 16\sqrt{2}$ |
| Reference | Cor. 1.1 | Cor. 1.1 | (1.5)* | (1.6) | (1.7) | (1.8) |

Before proceeding to the proof of Theorem 1.3, we would like to reflect again on the fact that $r_C$ is invariant under unitary similarities of C. We conclude, as in Theorem 4.2 of [5], that if $r_C$ has multiplicativity factors, then its optimal (smallest) factor $\mu_{r_C}$, is also unitarily invariant. It is easily seen, however, that while $\tau, \sigma, \lambda$ and $\varphi$ in (1.4) (which involve only the eigenvalues of C) are invariant under (unitary) similarities of C, the quantities $\omega, \rho$ and $\nu$ may well not be invariant. Hence, our lower bounds for $\mu$ in sections (iii) and (iv) of Theorem 1.3 are possibly not unitarily invariant, so in general these bounds are probably not optimal.

Although the bounds in (1.5) and (1.6) are unitarily invariant, we conjecture that usually they are far from optimal. The only instance in which we have knowingly achieved the best multiplicativity factor was the case of the classical numerical radius $r$, where we showed [5, Theorem 2.4] that $\mu_r = 4$;

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*If C is $2\times 2$ with eigenvalues of the same argument, then evidently $\varphi = \lambda$. Hence, the bound in (1.5) coincides with the one in (1.6), so there is no point in giving a $2\times 2$ example for Theorem 1.3 (i).

i.e., $\mu_r$ is a matrix norm on $\underset{\sim}{C}_{n\times n}$, $n \geq 2$, if and only if $\mu \geq 4$. As indicated in Theorem 2.4 of [5], this result holds for arbitrary (finite or infinite dimensional) Hilbert spaces, where the numerical radius of a bounded linear operator $A$ is

$$r(A) = \sup\{|(Ax,x)| : (x,x) = 1\} .$$

## 2. Proof of Theorem 1.3.

The main part of the proof consists of obtaining appropriate lower bounds for $r_C(A)$ in terms of the entries of $C$. We begin with,

LEMMA 2.1. [5, Lemma 4.1]. Let $C = (\gamma_{jk}) \in \underset{\sim}{C}_{n\times n}$ be an upper triangular matrix, and let $C_\ell$, $1 \leq \ell \leq n$, be the matrix obtained from $C$ by setting the off-diagonal entries in the last $n-\ell$ columns of $C$ equal to zero. Then for all $A \in \underset{\sim}{C}_{n\times n}$,

$$r_{C_\ell}(A) \leq r_{C_{\ell+1}}(A) \quad , \qquad \ell=1,\ldots,n-1 . \tag{2.1}$$

With this lemma we can easily prove:

LEMMA 2.2. Let $C = (\gamma_{jk}) \in \underset{\sim}{C}_{n\times n}$ be upper triangular with a diagonal part $D = \operatorname{diag}(\gamma_{11},\ldots,\gamma_{nn})$. Then

$$r_C(A) \geq r_D(A) \qquad \forall\ A \in \underset{\sim}{C}_{n\times n} .$$

Proof. Using the notation in Lemma 2.1 we have

$$C = C_n \ , \qquad D = C_1 .$$

Thus, by (2.1),

$$r_C(A) = r_{C_n}(A) \geq r_{C_{n-1}}(A) \geq \cdots \geq r_{C_1}(A) = r_D(A)$$

and the proof is complete. □

LEMMA 2.3. *If* $D = \text{diag}(\gamma_{11},\ldots,\gamma_{nn})$ *is a diagonal matrix, then*

$$r_D(A) \geq \frac{\tau}{n}\,|\text{tr}\,A| - (n-1)\,\delta r(A) \quad \forall\, A \in \underset{\sim}{C}_{n\times n} \quad ,$$

*where* $r(A)$ *is the classical numerical radius in* (1.2), *and* $\delta$ *and* $\tau$ *are defined in* (1.4).

*Proof:* We write

$$D = D_1 - D_2$$

where

$$D_1 = \left(\frac{1}{n}\,\text{tr}\,D\right)I \quad , \qquad D_2 = \text{diag}(\delta_1,\ldots,\delta_n) \quad ,$$

$$\delta_j = \frac{1}{n}\,\text{tr}\,D - \gamma_{jj} \ , \qquad j=1,\ldots,n \ .$$

Since a matrix $U \in \underset{\sim}{C}_{n\times n}$ is unitary if and only if its columns $u_1,\ldots,u_n$ are orthonormal (o.n.), we have

$$\begin{aligned}
(2.2) \qquad r_D &= \max\{|\text{tr}(DU^*AU)| : U \text{ unitary } n\times n\} \\
&= \max\{|\text{tr}(D_1U^*AU) - \text{tr}(D_2U^*AU)| : U \text{ unitary}\} \\
&= \max\left\{\left|\frac{1}{n}\,\text{tr}\,D\cdot\text{tr}\,A - \text{tr}(D_2U^*AU)\right| : U \text{ unitary}\right\} \\
&\geq \frac{\tau}{n}\,|\text{tr}\,A| - \max\{|\text{tr}(D_2U^*AU)| : U \text{ unitary}\} \\
&= \frac{\tau}{n}\,|\text{tr}\,A| - \max\left\{\left|\sum_{j=1}^{n}\delta_j u_j^*Au_j\right| : u_1,\ldots,u_n \text{ o.n.}\right\} \\
&\geq \frac{\tau}{n}|\text{tr}\,A| - \sum_{j=1}^{n}|\delta_j|\cdot\max\{|u^*Au| : u^*u = 1\} \\
&= \frac{\tau}{n}\,|\text{tr}\,A| - \sum_{j=1}^{n}|\delta_j|\,r(A) \ .
\end{aligned}$$

Now, writing for convenience $\gamma_{jj} = \gamma_j$,

$$(2.3)\quad |\delta_j| = |\gamma_j - \tfrac{1}{n}(\gamma_1 + \cdots + \gamma_n)|$$

$$= \frac{1}{n}\,|(\gamma_j - \gamma_1) + \cdots + (\gamma_j - \gamma_{j-1}) + (\gamma_j - \gamma_{j+1}) + \cdots + (\gamma_j - \gamma_r)|$$

$$\leq \frac{1}{n}\,\{|\gamma_j - \gamma_1| + \cdots + |\gamma_j - \gamma_{j-1}| + |\gamma_j - \gamma_{j+1}| + \cdots + |\gamma_j - \gamma_n|\}$$

$$\leq \frac{n-1}{n}\,\delta\ .$$

So by (2.2) and (2.3) the Lemma follows. □

REMARK. In the proof of Lemma 2.3 we have shown that

$$\sum_{j=1}^{n} |\delta_j| \leq (n-1)\delta\ .$$

It seems that this inequality can be improved to read

$$\sum_{j=1}^{n} |\delta_j| \leq n\delta/\sqrt{3}$$

This would follow from the following:

CONJECTURE. Given a set $S$ of $n$ points in Euclidean space so that the diameter of $S$ is $\delta$, then the sum of the distances of the points from the centroid of $S$ is maximal when the points are distributed in as nearly equal numbers as possible over the vertices of a regular simplex with edge-length $\delta$.

In the Euclidean plane this means the vertices of an equilateral triangle, so that for $n$ which is a multiple of 3 we get exactly $n$ distances $\delta/\sqrt{3}$ from the centroid of the triangle.

Having Lemmas 2.2 and 2.3, we immediately obtain:

COROLLARY 2.1. _Let_ $C \in \underset{\sim}{C}_{n\times n}$ _be upper triangular, and for_ $A \in \underset{\sim}{C}_{n\times n}$ _let_ $\theta$ _satisfy_

$$|\mathrm{tr}\ A| = \theta n\ r(A) .$$

_Then_

$$r_C(A) \geq \{\tau\theta - (n-1)\delta\} r(A) .$$

We turn now to study the contribution of the off-diagonal entries of $C$ to $r_C(A)$.

LEMMA 2.4. _Let_ $C = (\gamma_{jk}) \in \underset{\sim}{C}_{n\times n}$ _be upper triangular, and let_ $\gamma \geq 0$ _be the largest absolute value of the off-diagonal entries of_ $C$ _as defined in_ (1.4). _Then_

$$r_C(A) \geq \gamma \cdot R(A) \qquad \forall\ A \in \underset{\sim}{C}_{n\times n} ,$$

_where_

$$R(A) \equiv \max\{|x^*Ay| : x,y \text{ o.n. } \underline{\text{in}}\ \underset{\sim}{C}^n\} .$$

_Proof_. Let $\gamma_{pq}$, $p < q$, be an off-diagonal element of $C$ satisfying

$$|\gamma_{pq}| = \gamma , \tag{2.4}$$

and let $C_q$ be the matrix obtained from $C$ by setting all off-diagonal entries in the last $n-q$ columns of $C$ equal to zero.

Since for any $B = (\beta_{jk}) \in \underset{\sim}{C}_{n\times n}$ we have

$$\begin{aligned} r_B(A) &= \max\{|\mathrm{tr}\,(BU^*AU)| : U \text{ unitary}\} \\ &= \max\left\{\left|\sum_{j,k} \beta_{jk}\ u_k^*Au_j\right| : u_1,\ldots,u_n \text{ o.n.}\right\} , \end{aligned}$$

then

$$r_{C_q}(A) = \max\left\{\left|\sum_{j=1}^{n} \gamma_{jj}\ u_j^*Au_j + \sum_{j<k\leq q} \gamma_{jk}\ u_k^*Au_j\right| : u_1,\ldots,u_n \text{ o.n}\right\} . \tag{2.5}$$

Now let $v_1,\ldots,v_n \in \underset{\sim}{C}^n$ be an o.n. system such that

$$|v_q^* A v_p| = R(A) \quad ,$$

and denote

$$\Omega_{pq} \equiv \arg(\gamma_{pq}\, v_q^* A v_p) \quad ,$$

$$\Omega_q \equiv \arg\left(\sum_{\substack{j=1\\ j\neq p}}^{q-1} \gamma_{jq}\, v_q^* A v_j\right) \quad ,$$

$$w_j \equiv \begin{cases} v_j & , \quad j \neq p \\ v_j\, e^{i(\Omega_q - \Omega_{pq})} & , \quad j = p \quad . \end{cases}$$

Then $w_1,\ldots,w_n$ are o.n. with

$$|w_q^* A w_p| = R(A) \quad ,$$

and

$$\arg(\gamma_{pq}\, w_q^* A w_p) = \arg\left(\sum_{\substack{j=1\\ j\neq p}}^{q-1} \gamma_{jq}\, w_q^* A w_j\right) = \Omega_q$$

Next, denote

$$\Omega \equiv \arg\left(\sum_{j=1}^{n} \gamma_{jj} w_j^* A w_j + \sum_{j<k\leq q-1} \gamma_{jk}\, w_k^* A w_j\right) \quad ,$$

$$z_j = \begin{cases} w_j & , \quad j \neq q \\ w_j\, e^{i(\Omega_q - \Omega)} & , \quad j = q \quad . \end{cases}$$

So now $z_1,\ldots,z_n$ are o.n. with

(2.6) $$|z_q^* A z_p| = R(A)$$

and

$$(2.7)\quad \arg(\gamma_{pq}\ z_q^* A z_p) = \arg\left(\sum_{\substack{j=1\\ j\neq p}}^{q-1} \gamma_{jq}\ z_q^* A z_j\right)$$

$$= \arg\left(\sum_{j=1}^{n} \gamma_{jj}\ z_j^* A z_j + \sum_{j<k\leq q-1} \gamma_{jk}\ z_k^* A z_j\right) = \Omega\ .$$

By (2.5) - (2.7) and Lemma 2.1, therefore,

$$\gamma \cdot R(A) = \gamma|\gamma_{pq}\ z_q^* A z_p| \leq \left|\sum_{j=1}^{n} \gamma_{jj}\ z_j^* A z_j + \sum_{j<k\leq q} \gamma_{jk}\ z_k^* A z_j\right|$$

$$\leq r_{C_q}(A) \leq r_{C_{q+1}}(A) \leq \cdots \leq r_{C_n}(A) = r(A)\ . \qquad \square$$

We now quote an interesting result of Stolov.

LEMMA 2.5. [10, *Theorem* 2]. *For any* $A \in \underset{\sim}{C}_{n\times n}$,

$$R(A) \geq \text{rad } W(A)\ ,$$

*where* rad $W(A)$ *is the circumradius of the numerical range of* $A$, *i.e., the radius of the smallest disc containing the set*

$$W(A) = \{x^* A x : x \in \underset{\sim}{C}^n,\ x^* x = 1\}\ .$$

Next we prove,

LEMMA 2.6. *For any* $A \in \underset{\sim}{C}_{n\times n}$,

$$\text{rad } W(A) \geq \frac{1}{2}\, r(A) - \frac{1}{2n}\, |\text{tr } A|\ .$$

*Proof*. Since the Toeplitz-Hausdorff Theorem (e.g.[2,7]) states that the numerical range is a convex set and since the eigenvalues of $A$ are contained in $W(A)$ (again [2, 7]), then the centroid of these eigenvalues, $(1/n)\text{tr } A$, is a point in $W(A)$. Consequently,

$$r(A) = \max\{|\zeta| : \zeta \in W(A)\} \leq \frac{1}{n} |\mathrm{tr}\ A| + 2\ \mathrm{rad}\ W(A) \quad ,$$

and the lemma follows. □

Our three last lemmas lead to:

COROLLARY 2.2. *Let* $A \in \underset{\sim}{C}_{n\times n}$ *be given and let* $\theta$ *be determined by*

$$|\mathrm{tr}\ A| = \theta n\ r(A) \quad .$$

*Then*

$$r_C(A) \geq \frac{\gamma}{2}(1-\theta)\ r(A) \quad .$$

*Proof*. By Lemma 2.4 - 2.6, we have

$$r_C(A) \geq \gamma \cdot R(A) \geq \gamma \cdot \mathrm{rad}\ W(A)$$

$$\geq \frac{\gamma}{2} r(A) - \frac{\gamma}{2n} |\mathrm{tr}\ A| = \frac{\gamma}{2}(1-\theta)\ r(A) \quad . \ \square$$

We are now able to obtain the following lower bound for $r_C(A)$.

LEMMA 2.7. *Let* $C = (\gamma_{jk}) \in \underset{\sim}{C}_{n\times n}$ *be upper triangular with* $\mathrm{tr}\ C \neq 0$.

*Then*:

(i) *For all* $A \in \underset{\sim}{C}_{n\times n}$,

(2.8) $$r_C(A) \geq \xi \|A\|_2 \quad ,$$

*where*

$$\xi \equiv \frac{\tau\gamma - (n-1)\delta\gamma}{4\tau + 2\gamma} \quad ;$$

$\tau, \delta$ *and* $\gamma$ *are as in* (1.4); *and*

$$\|A\|_2 = \max\{(x^*A^*Ax)^{1/2} : x \in \underset{\sim}{C}^n, \quad x^*x = 1\}$$

*is the spectral (i.e.* $\ell^2$) *norm of* A.

(ii) <u>If in addition, the eigenvalues of</u> $C$ <u>are equal, then for all</u> $A \in \underset{\sim}{C}_{n\times n}$,

$$r_C(A) \geq \rho \|A\|_2$$

<u>where</u> $\rho$ <u>is defined in</u> (1.4).

<u>Proof</u>. Take any $A \in \underset{\sim}{C}_{n\times n}$ and let $\theta$ satisfy $|\mathrm{tr}\, A| = \theta n\, r(A)$. Then by Corollaries 2.1 and 2.2,

$$r_C(A) \geq \max\{\tau\theta - (n-1)\delta, \tfrac{\gamma}{2}(1-\theta)\}\, r(A) \quad . \tag{2.9}$$

Since $\gamma \geq 0$ and $\tau = |\mathrm{tr}\, C| > 0$, then the expressions in the braces are functions of $\theta$ describing straight lines with opposite slopes which intersect for $\theta = \theta_0$ where

$$\theta_0 = \frac{\gamma + 2(n-1)\delta}{2\tau + \delta} \quad .$$

Thus, for any $\theta$,

$$\max\{\tau\theta - (n-1)\delta, \tfrac{\gamma}{2}(1-\theta)\} \geq \tfrac{\gamma}{2}(1-\theta_0) = 2\xi \quad ; \tag{2.10}$$

and (2.9), (2.10) yield

$$r_C(A) \geq 2\xi\, r(A) \quad .$$

This together with the well known relation (e.g. [6, 7])

$$r(A) \geq \tfrac{1}{2} \|A\|_2$$

gives (2.8).

Part (ii) of the lemma follows from the fact that if the eigenvalues of $C$ are equal, then $\delta = 0$ and $\xi = \rho$. □

The lower bound for $r_C(A)$ in (2.8) vanishes as the off-diagonal entries of $C$ vanish and we are interested now in bounds which depend only on the eigenvalues. This was done in [5] as described by our next lemma which holds for matrices $C$ that need not be triangular.

LEMMA 2.8. *Let* $C \in \underset{\sim}{C}_{n\times n}$ *have eigenvalues* $\gamma_1,\ldots,\gamma_n$ *and let* $\operatorname{tr} C \neq 0$. *Then:*

(i) *For all* $A \in \underset{\sim}{C}_{n\times n}$,

$$r_C(A) \geq \lambda \|A\|_2 \quad , \tag{2.11}$$

*where in accordance with* (1.4),

$$\lambda = \frac{\tau\delta}{4(1-1/n)\tau + 2\delta}\ , \qquad \delta = \max_{j,k}|\gamma_j - \gamma_k|, \qquad \tau = |\operatorname{tr} C| = |\sum_{j=1}^{n}\gamma_j| \quad .$$

(ii) *If* $C$ *is normal with eigenvalues of the same argument, then for all* $A \in \underset{\sim}{C}_{n\times n}$,

$$r_C(A) \geq \varphi \|A\|_2 \tag{2.12}$$

*with* $\varphi$ *as defined in* (1.4).

*Proof*. By Lemma 4.2 of [5], if $\kappa = \kappa(\gamma_1,\ldots,\gamma_n)$ satisfies the inequality (3.1) of [5], then

$$r_C(A) \geq \frac{\kappa}{2}\|A\|_2 \qquad \forall\ A \in \underset{\sim}{C}_{n\times n} \quad . \tag{2.13}$$

Reviewing the proof of Theorem 3.1 (ii) of [5] we find without difficulty that since $\tau = |\operatorname{tr} C| > 0$ ($\delta$ may vanish), then $\kappa = \tau\delta/(2\tau - 2\tau/n + \delta)$ satisfies inequality (3.1) of [5]; so (2.13) implies (2.11).

For part (ii) of the lemma, we mention that by Theorem 3.1 (iii) of [5], if the $\gamma_j$ are of the same argument, then inequality (3.1) of [5] holds with $\tau = \delta/2$. Hence (2.13) yields

$$r_C(A) \geq \frac{\delta}{4}\|A\|_2$$

which, if combined with (2.11), gives (2.12).

Our next results provides upper bounds for $r_C(A)$.

LEMMA 2.9. [5, Lemma 4.3]. Let $C = (\gamma_{jk}) \in \mathbb{C}_{n\times n}$ have eigenvalues $\gamma_1,\ldots,\gamma_n$. Set

$$\omega = \min\left\{\sum_{j=1}^{n}\sqrt{\sum_{k=1}^{n}|\gamma_{jk}|^2}\ ,\quad \sum_{k=1}^{n}\sqrt{\sum_{j=1}^{n}|\gamma_{jk}|^2}\right\},\quad \sigma = \sum_{j=1}^{n}|\gamma_j|,\ \tau = \left|\sum_{j=1}^{n}\gamma_j\right|$$

(which agrees with (1.4) if $C$ is triangular.) Then:

(i) $$r_C(A) \le \omega\|A\|_2 \qquad \forall\ A \in \mathbb{C}_{n\times n}\ .$$

(ii) For normal $C$

$$r_C(A) \le \sigma\|A\|_2 \qquad \forall\ A \in \mathbb{C}_{n\times n}\ .$$

(iii) For normal $C$ with eigenvalues of the same argument,

$$r_C(A) \le \tau\|A\|_2 \equiv \sigma\|A\|_2 \qquad \forall\ A \in \mathbb{C}_{n\times n}\ .$$

Proof. The proof of (i) is given in [5]. Parts (ii) and (iii), whose proof was omitted by mistake, follow immediately from part (i) and from the fact that since $r_C$ is invariant under unitary similarities of $C$, then for normal $C$ we may take $C = \mathrm{diag}(\gamma_1,\ldots,\gamma_n)$. □

We still need the following version of a result of Gastinel.

LEMMA 2.10 ([1], [4, Theorem 5].) Let $M$ and $N$ be a matrix norm and a generalized matrix norm on $\mathbb{C}_{n\times n}$, respectively; and let $\eta \ge \zeta > 0$ be constants satisfying

$$\zeta M(A) \le N(A) \le \eta\, M(A) \qquad \forall\ A \in \mathbb{C}_{n\times n}\ .$$

Then any $\mu$ with $\mu \geq \eta/\zeta^2$ is a multiplicativity factor for $N$.

With Lemmas 2.7 - 2.10 we are finally ready for:

Proof of Theorem 1.3. (i) If $C$ is normal with eigenvalues of equal argument and $\text{tr}\, C \neq 0$, then by Lemmas 2.8 (ii) and 2.9 (iii),

$$\varphi\|A\|_2 \leq r_C(A) \leq \tau\|A\| \equiv \sigma\|A\|_2 \qquad \forall\ A \in \underset{\sim}{C}_{n\times n} .$$

Since $C$ is diagonal but not scalar, we have $\delta > 0$. Thus $\varphi > 0$, so Lemma 2.10 holds with

$$M = \|\cdot\|_2, \qquad N = r_C, \qquad \eta = \tau = \sigma, \qquad \zeta = \varphi ,$$

and (1.5) follows.

(ii) If $C$ is normal with $\text{tr}\, C \neq 0$, then Lemmas 2.8 (i) and 2.9 (ii) give

$$\lambda\|A\|_2 \leq r_C(A) \leq \sigma\|A\|_2 \qquad \forall\ A \in \underset{\sim}{C}_{n\times n} .$$

Since the eigenvalues are not all equal and $\text{tr}\, C \neq 0$ then $\tau > 0$ and $\delta > 0$; so $\lambda > 0$, and Lemma 2.10 again implies (1.6).

(iii) By Lemmas 2.7 (ii) and 2.9 (i),

$$\rho\|A\|_2 \leq r_C(A) \leq \omega\|A\|_2 \qquad \forall\ A \in \underset{\sim}{C}_{n\times n} .$$

Again $\tau = |\text{tr}\, C| > 0$, and since $C$ is nonnormal then $\gamma > 0$ too. Thus, $\rho > 0$, and Lemma 2.10 implies (1.7).

(iv) Last, if $C$ is nonnormal with eigenvalues not all equal, then by Lemmas 2.7 (i), 2.8 (i) and 2.9 (i) we have

$$\nu\|A\|_2 \equiv \max\{\xi,\lambda\}\cdot\|A\|_2 \leq r_C(A) \leq \omega\|A\|_2 \qquad \forall\ A \in \underset{\sim}{C}_{n\times n} .$$

As in part (i), $\lambda > 0$; so $\nu > 0$, and Lemma 2.10 completes the proof.

$\square$

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For two n-square matrices $A, C$, the C-numerical radius of $A$ is the nonnegative quantity

$$r_C(A) = \max\{|\operatorname{tr}(CU^*AU)| : U \text{ unitary}\}.$$

This generalizes the classical numerical radius $r(A)$. It is known that

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ITEM #20, CONTINUED:

$r_C$ constitutes a norm on the algebra of n-square matrices if and only if $C$ is nonscalar and $\operatorname{tr} C \neq 0$. For all such $C$ we obtain multiplicativity factors for $r_C$, i.e., constants $a > 0$ for which $ar_C$ is sub-multiplicative.